Math 245C Lecture 5 Notes

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1 The Marcinkiewicz Interpolation Theorem (cont.)

Today's lecture was given by a guest lecturer, Alpár Mészáros.

1.1 Continuation of the proof

Last time, we were proving the Marcinkiewicz interpolation theorem.

Theorem 1.1 (Marcinkiewicz interpolation theorem)**.** *Let $\mathcal{F}$ be the set of measurable functions on Y. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ be real numbers such that $p_0 \leq q_0$, $p_1 \leq q_1$, and $q_0 \neq q_1$. Let $t \in (0,1)$, and let p, q be defined as*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Assume that $T : L^{p_0}(\mu) + L^{p_1}(\mu) \to \mathcal{F}$ be sublinear and of weak type (q_0, p_0) and (q_1, p_1) (there are $c_0, c_1 > 0$ such that if $q_0, q_1 \neq \infty$, $(\alpha^{q_0} \lambda_{T(f)})^{1/q_0} \leq c_0 \|f\|_{p_0}$ and $(\alpha^{q_1} \lambda_{T(f)})^{1/q_1} \leq c_1 \|f\|_{p_1}$). Then the following hold:

1. *T is strong type (p, q) (there exists $B_p > 0$ such that $\|Tf\|_q \leq B_p \|f\|_p$ for all $f \in L^p(\mu)$).*
2. *If $p_0 < \infty$, then $\lim_{p \to p_0} B_p |p_0 - p| < \infty$. If $p_1 < \infty$, then $\lim_{p \to p_1} B_p |p_1 - p| < \infty$. If $p_0 = \infty$, (B_p) remains bounded as $p \to p_0$. If $p_1 = \infty$, (B_p) remains bounded as $p \to p_1$.*

Proof. The general idea is the decompose the function f into two parts: for $A > 0$, cut off the function f if it exceeds A. So if $E(A) = \{x : |f(x)| > A\}$, we define $h_A = f \mathbb{1}_{X \setminus E(A)} + A \mathbb{1}_{E(A)}$ and $g_A = f - h_A$. First assume $q_0 \neq q_1$, and assume $q_0, q_1 < \infty$. Take q as in the theorem. If $f \in L^{p_0} + L^{p_1}$, then

$$\|Tf\|_q^q = q \int_0^\infty \alpha^{q-1} \lambda_{Tf}(\alpha) \, d\alpha$$

Since T is sublinear, we have $\lambda_{Tf}(2\alpha) \leq \lambda_{Tg_A}(\alpha) + \lambda_{Th_A}(\alpha)$ for all $\alpha, A > 0$ (independently of each other). We get, after a change of variables,

$$\|Tf\|_q^q \leq q2^q \int_0^\infty \alpha^{q-1}\lambda_{Tf}(2\alpha)\, d\alpha \leq 2^q q \underbrace{\int_0^\infty \alpha^{q-1}\lambda_{Th_A}(\alpha)}_{=I_1} + \underbrace{\alpha^{q-1}\lambda_{Tg_A}(\alpha)\, d\alpha}_{=I_2}.$$

Look at I_2:

$$\begin{aligned}
I_2 &= 2^q q \int_0^\infty \alpha^{q-1}\frac{\alpha^{q_0}}{\alpha^{q_0}}\lambda_{Tg_A}(\alpha)\, d\alpha \\
&\leq 2^q q \int_0^\infty \alpha^{q-q_0-1}[Tg_A]_{q_0}^{q_0}\, d\alpha \\
&\leq 2^q q \int_0^\infty \alpha^{q-q_0-1}(c_0\|g_A\|_{p_0})^{q_0}\, d\alpha \\
&= 2^q q C_0^{q_0} \int_0^\infty \alpha^{q-q_0-1}\|g_a\|_{p_0}^{q_0}\, \alpha.
\end{aligned}$$

Now

$$\begin{aligned}
\|g_A\|_{p_0}^{p_0} &= p_0 \int_0^\infty \alpha^{p_0-1}\lambda_{h_A}(\alpha)\, d\alpha \\
&= p_0 \int_0^\infty \alpha^{p_0-1}\lambda_f(\alpha + A)\, d\alpha \\
&= p_0 \int_A^\infty (\alpha - A)^{p_0-1}\lambda_f(\alpha)\, d\alpha \\
&\leq p_0 \int_A^\infty \alpha^{p_0-1}\lambda_f(\alpha)\, d\alpha
\end{aligned}$$

$$\|h_A\|_{p_0}^{p_0} = p_0 \int_0^\infty \alpha^{q_0-1}\lambda_{h_A}(\alpha)\, d\alpha = p_0 \int_0^A \alpha^{p_0-1}\lambda_f(\alpha)\, d\alpha$$

Combing back to $\|Tf\|_q^q$, we get

$$\begin{aligned}
\|Tf\|_q^q &\leq 2^q q C_0^{q_0} \int_0^\infty \alpha^{q-q_0-1}\|g_A\|_{p_0}^{q_0}\, d\alpha + 2^q q C_1^{q_1} \int_0^\infty \alpha^{q-q_1-1}\|h_A\|_{p_1}^{q_1} \\
&\leq 2^q q C_0^{q_0} \int_0^\infty \alpha^{q-q_0-1}\left(p_0 \int_A^\infty \beta^{p_0-1}\lambda_f(\beta)\, d\beta\right)^{q_0/p_0} d\alpha \\
&\qquad + 2^q q C_1^{q_1} \int_0^\infty \alpha^{q-q_1-1}\left(p_1 \int_0^A \beta^{p_1-1}\lambda_f(\beta)\, d\beta\right)^{q_1/p_1} d\alpha \\
&= \sum_{j=0}^{1} 2^q q C_j^{q_j} p_j^{q_j/p_j} \int_0^\infty \left(\int_0^\infty \phi(\alpha, \beta)\, d\beta\right) d\alpha,
\end{aligned}$$

where

$$\phi(\alpha, \beta) := \mathbb{1}_j(\alpha, \beta)\beta^{p_j - 1}\lambda_f(\beta)\alpha^{(q - q_j - 1)p_j/q_j},$$

$\mathbb{1}_0$ is the indicator of $\{(\alpha, \beta) : \beta > A\}$, and $\mathbb{1}_1$ is the indicator of $\{(\alpha, \beta) : \beta < A\}$.

In remains to study the terms separately with a special choice of A. Using Minkowski's inequality,

$$\int_0^\infty \left(\int_0^\infty \phi_j(\alpha, \beta)\, d\beta \right)^{q_j/p_j} d\alpha \le \left(\int_0^\infty \left(\int_0^\infty \phi_j(\alpha, \beta)^{q_j/p_j}\, d\beta \right)^{p_j/q_j} d\alpha \right)^{q_j/p_j}$$

Choose $\sigma > 0$ and set $A = \alpha^\sigma$. Then $\alpha \le \beta^{1/\sigma}$. The inside of the above integral for $j = 0$ is (for a special choice of σ),

$$\begin{aligned}
\int_0^\infty \left(\int_0^{\beta^{1/\sigma}} \alpha^{q - q_0 - 1}\, d\alpha \right)^{p_0/q_0} \beta^{p_0 - 1}\lambda_f(\beta)\, d\beta &= \int_0^\infty \frac{1}{q - q_0} \left([\alpha]_0^{\beta^{1/\sigma}} \right)^{p_0/q_0} \beta^{p_0 - 1}\lambda_f(\beta)\, d\beta \\
&= (q - q_0)^{-p_0/q_0} \int_0^\infty \beta^{p_0 - 1 + (q - q_0)/\sigma}\lambda_f(\beta)\, d\beta \\
&= (q - q_0)^{-p_0/q_0} \int_0^\infty \beta^{p - 1}\lambda_f(\beta)\, d\beta \\
&= (q - q_0)^{-p_0/q_0} p^{-1} \|f\|_p^p.
\end{aligned}$$

The other term is similar. We will finish the proof next time. □